THE
MATHEMATICAL GAZETTE.

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THE SOUTHAMPTON BRANCH OF THE MATHEMATICAL ASSOCIATION.

THREE meetings were held in King Edward VI. Grammar School during the first half session of 1912.

In the first held on January 26th, Mr. W. T. Tregear, B.A., Headmaster of the County School, Sandown, Isle of Wight, gave an account of some experiments in teaching made by himself with classes of boys and girls. Mr. Tregear was led to the conclusion that too rapid progress could not be expected in this time of transition, and that the syllabus of Mathematics must not be very wide.

The meeting was unanimous in thinking that children required a formal framework into which to fit their thoughts when writing out propositions in Geometry.

In the second meeting, held on February 23rd, Mr. W. D. Evans, M.A., of the Hartley College, read an abstruse paper on the "Principles of Mechanics." He treated of the Ideas of Mass, Force and Acceleration as enumerated in the Text-Books, and as set forth by Sir I. Newton. He endeavoured to build up the Idea of Mass from the consideration of Accelerations, which themselves were derived from a reference of Forces to axes fixed in space. The lecturer acknowledged his indebtedness to Mach.

In the third meeting, held on March 22nd, Mr. J. B. Crompton, M.A., of King Edward VI. School, read a paper on the "Teaching of Mechanics."

Mr. Crompton said there were three points of departure for the teacher: (1) The Lever, (2) The Principle of Virtual Work, (3) The Parallelogram of Forces. For himself he began with the last, which he approached by means of experimental work with spring balances. He considered heuristic methods unsuitable for teaching elementary mechanics, but said experiments could be devised to illustrate the chief Statical Laws. Dynamical experiments were also possible, but more difficult to manage.

A long and animated discussion took place, which was concerned chiefly with side issues such as the various proofs of the parallelogram of forces, and some points in the theory of work and moments.

The Mathematical Association.

LONDON BRANCH.

PRESIDENTIAL ADDRESS ON THE THEORY OF PROPORTION.

By Professor M. J. M. Hill, M.A., LL.D., Sc.D., F.R.S.

10*th February*, 1912. (*Continued.*)

IDEAS LEADING UP TO EUCLID'S FIFTH AND SEVENTH DEFINITIONS.

16. In the second edition of my *Contents of the Fifth and Sixth Books of Euclid* I have taken the ideas stated in the preceding article as my starting point, and built up the conditions involved in the 5th and 7th definitions in the following way:

1. The ratio of a magnitude A to another magnitude B *of the same kind* is a real number, rational or irrational, determined in the manner explained in what follows. It is denoted by the symbol $A : B$.

2. If the magnitudes A and B have a common measure G, so that $A = aG$, $B = bG$, where a, b are positive whole numbers, then $A : B$ is defined to be the rational number $\frac{a}{b}$.

Using this definition it can be proved that, if C be another magnitude, which is also measured by G, so that $C = cG$, where c is a positive whole number,

$$A : C \gtreqless B : C,$$

according as

$$A \gtreqless B,$$

and conversely.

3. If A and B have no common measure, then the definition given in the preceding article is inapplicable, and it is necessary to say either that A cannot have a ratio to B or else to give a new definition of the ratio of A to B.

This new definition must be of such a nature that when it is applied to two magnitudes A and B which have a common measure, it will give the same value for the ratio of A to B as that which would be given by the definition of the preceding section. (Otherwise there would be a conflict between the two definitions.)

This second alternative will be chosen.

4. Proceeding on these lines the following fundamental assumptions are made, it being supposed that A, B, C are magnitudes *of the same kind*; they may, or they may not, have a common measure:

   (i) If A be greater than B, then the ratio of A to C is greater than that of B to C.

   (ii) If A be equal to B, then the ratio of A to C is equal to that of B to C.

To these may be added for symmetry an assumption which is included in (i).

   (iii) If A be less than B, then the ratio of A to C is less than that of B to C.

These statements may be abbreviated into the following:

(i) If $A > B$, then $A : C > B : C$;

(ii) If $A = B$, then $A : C = B : C$;

(iii) If $A < B$, then $A : C < B : C$.

It is obvious that between these assumptions and the results stated in (2) there can be no conflict.

5. From the fundamental assumptions in (4) it follows as a logical consequence that

(iv) If $A : C > B : C$, then $A > B$;

(v) If $A : C = B : C$, then $A = B$;

(vi) If $A : C < B : C$, then $A < B$.

6. It can now be proved that

(i) If $rA > nB$, then $A : B > \frac{n}{r}$,

(ii) If $rA = nB$, then $A : B = \frac{n}{r}$,

(iii) If $rA < nB$, then $A : B < \frac{n}{r}$.

These results make it possible to determine the position of the ratio $A : B$ in the system of rational numbers.

It is therefore regarded as a known number. It is known in the sense that all its properties can be determined from a knowledge of its position in the system of rational numbers.

The converse propositions to (i), (ii), (iii) can now be proved.

(iv) If $A : B > \frac{n}{r}$, then $rA > nB$,

(v) If $A : B = \frac{n}{r}$, then $rA = nB$,

(vi) If $A : B < \frac{n}{r}$, then $rA < nB$.

7. Two ratios are defined as *equal* when no rational number whatever can lie between them.

From this definition the conditions of Euclid's Fifth Definition can be deduced.

8. Two ratios will be said to be *unequal* when any single rational number can be shown to lie between them.

This is equivalent to (but differs somewhat in form from) Euclid's Seventh Definition.

DEDUCTION OF THE CONDITIONS OF THE FIFTH DEFINITION.

17. Let the ratios be $A : B$ and $C : D$.

Then if they be equal, no rational fraction can lie between them (Art. 16 (7)).

Let $\frac{s}{r}$ be any rational fraction.

Comparing it with $A : B$, one of the following alternatives must hold:

(i) $A : B > \frac{s}{r}$, or (ii) $A : B = \frac{s}{r}$, or (iii) $A : B < \frac{s}{r}$.

(i) If $A : B > \frac{s}{r}$, then, in order that $\frac{s}{r}$ may not lie between $A : B$ and $C : D$, $C : D$ must also be greater than $\frac{s}{r}$.

Now if $A : B > \frac{s}{r}$, then $rA > sB$ and conversely.

And if $C : D > \frac{s}{r}$, then $rC > sD$ and conversely.

Hence, if $rA > sB$, then $rC > sD$.

(ii) If $A : B = \frac{s}{r}$, and if no rational fraction lie between $A : B$ and $C : D$, then must $C : D = \frac{s}{r}$.

But if $A : B = \frac{s}{r}$, then $rA = sB$ and conversely; and if $C : D = \frac{s}{r}$, then $rC = sD$ and conversely.

Hence, if $rA = sB$, then $rC = sD$.

(iii) Similarly, if $A : B < \frac{s}{r}$, and therefore $C : D < \frac{s}{r}$, it can be shown that here $rA < sB$, and then $rC < sD$.

Hence, if two ratios are defined to be equal when no rational fraction lies between them, the magnitudes appearing in the ratios satisfy the conditions of Euclid's Fifth Definition.

18. Before proceeding to deduce the Seventh Definition it will be proved that if two ratios are unequal according to the first and third of the Fundamental Assumptions in Art. 16 (4), then a rational fraction must lie between them.

It is there assumed as fundamental that, if A, B, C be magnitudes of the same kind, and if $A > B$, then $A : C > B : C$.

Now, if $A > B$, then $A - B$ is a magnitude of the same kind as A, B and C.

Hence, by Archimedes' Axiom, an integer n exists, such that

$$n(A - B) > C; \qquad \therefore\ nA > nB + C.$$

It can now be shown (see Art. 24 below) that an integer t exists, such that

$$nA > tC > nB; \qquad \therefore\ A : C > \frac{t}{n} > B : C.$$

If therefore two ratios are unequal under the circumstances arising out of the fundamental assumptions in Art. 16, then a rational fraction lies between them.

If therefore ratios are said to be unequal when a rational fraction lies between them, this will not be inconsistent with the fundamental assumption that if $A > B$, then $A : C > B : C$.*

Deduction of the Seventh Definition.

19. Two ratios are defined to be unequal when a rational fraction lies between them. Art. 16 (8).

Suppose that the rational fraction $\frac{s}{r}$ lies between $A : B$ and $C : D$, and suppose that $A : B > C : D$, then $A : B > \frac{s}{r} > C : D$;

$$\therefore\ rA > sB, \text{ but } rC < sD.$$

This is not exactly the form which Euclid gives to his Seventh Definition. He says that $A : B > C : D$ if integers r, s can be found such that when $rA > sB$, either $rC < sD$ or $rC = sD$. And as Saccheri and Commandinus remark (Heath, *l.c.* Vol. II. p. 130), he might also have given the condition

$$rA = sB, \text{ but } rC < sD.$$

* I am indebted to Mr. Rose-Innes for the introduction of this article at this stage of the argument.

It can be shown that in the cases

(i) $rA = sB$, $rC < sD$ and (ii) $rA > sB$, $rC = sD$,

other integers r', s' exist, such that $r'A > s'B$, $r'C < s'D$.

It will be enough to take the first case.

Since $rC < sD$,

$\therefore$ $sD - rC$ is a magnitude of the same kind as C.

Hence an integer n exists, such that

$$n(sD - rC) > C; \qquad \therefore (nr+1)C < nsD.$$

But since $rA = sB$,

$$\therefore nrA = nsB; \qquad \therefore (nr+1)A > nsB.$$

If $\therefore$ we replace $nr+1$ by r', ns by s', we shall have

$$r'A > s'B, \text{ but } r'C < s'D.$$

It is therefore sufficient to use the condition that $rA > sB$, $rC < sD$, the one directly suggested by the definition here given.

The Sufficiency of the Fifth Definition to Prove all the Properties of Equal Ratios.

20. The conditions set out in the 5th and 7th Definitions have now been reached as a consequence of certain fundamental assumptions and definitions.*

Whether Euclid reached his definitions in this or some other way it is of course impossible to determine. But I feel able to assert that with such explanations as are given above, it is possible to explain the argument to persons of average mathematical capacity.

It should be noticed that in building up the Fifth Definition (the Test for Equal Ratios), though the consideration of such inequalities as $A : B > \frac{n}{r}$ or $A : B < \frac{n}{r}$ is necessary, yet the Seventh Definition (the Test for distinguishing between Unequal Ratios) is not employed.

The Fifth Definition is therefore absolutely distinct from the Seventh Definition.

Hence arguing a priori *it might be expected that it would prove to be possible to obtain all the properties of Equal Ratios by means of the Fifth Definition only, if that definition is a full and complete one. This is in fact the case.*

With this idea to guide one, the proofs of most of the properties of Equal Ratios may be conducted on nearly the same lines.

Thus, if it is desired to prove that, under certain conditions, $A : B = C : D$, the first step is to take any rational fraction $\frac{s}{r}$. Compare it with $A : B$, then one of the following alternatives must hold:

(i) $A : B > \frac{s}{r}$, or (ii) $A : B = \frac{s}{r}$, or (iii) $A : B < \frac{s}{r}$.

Suppose $A : B > \frac{s}{r}$, then $rA > sB$.

Now, on making use of the given conditions, it follows very readily *in most cases* that $rC > sD$, and $\therefore$ $C : D > \frac{s}{r}$.

Similarly, if $A : B < \frac{s}{r}$, then $C : D < \frac{s}{r}$.

* It may be noticed that Euclid does not in his Fifth Book assume, as has been done here, the divisibility of any magnitude into any number of equal parts, an assumption which is required for the proofs of the propositions in Art. 16 (6) (see my *Euclid V. and VI.*, 2nd Edition, Art. 44). He does, however, make this assumption in the 6th Proposition of the Tenth Book (Heath, *l.c.* Vol. III. p. 26).

It is not necessary to consider the second alternative (see Art. 14);

$\therefore$ no rational fraction falls between $A : B$ and $C : D$.

$\therefore A : B = C : D$.

To the statement that what I have described can be done "in most cases very readily," it is necessary to add that in the case of three important propositions, Nos. 16, 22 and 23, the work is greatly facilitated by using the theorem that if $A > B$ and if C be a magnitude of the same kind as A and B, then integers n, t exist, such that $nA > tC > nB$ (see Art. 24). With the aid of this proposition, the proofs of these three propositions are very similar to one another.

On the other hand, Propositions 19, 24 and 25 cannot be easily obtained directly from the Fifth Definition. It is most convenient to deduce 19 from 16 and 17, 24 from 18 and 22, and 25 from 16 and 17, much as Euclid does.

If, on the other hand, the argument is conducted on Euclid's lines, then many of the properties of Equal Ratios are obtained by direct or indirect use of the Seventh Definition and Propositions 8, 10 and 13, which depend on it, so that the properties of *Equal* Ratios given in Props. 9, 14, 16 and 18-25 are proved by the aid of properties of *Unequal* Ratios.

This makes the proofs difficult and intricate; they do not appear to run on similar lines. It is not obvious when the 5th definition is to be used, and when the 7th definition or one of the propositions depending on it is to be used. And it is very difficult even for those who are able to grasp the several steps of the argument when taken singly to understand the argument as a whole.

I am therefore fully prepared to admit that it is quite useless to include Euclid's argument in any course of elementary instruction. It should be noted that what I propose is to assimilate the proofs of the Fourth group of propositions to those given by Euclid of the Second group. My proposal cannot therefore be objected to as a departure from Euclid's spirit. I think that the explanation on the lines here proposed should form a part of the mathematical education of those who have obtained a knowledge of elementary algebra and the contents of Euclid's first four books, and who desire to proceed further, because the Fifth Book and especially the Fifth Definition are the source from which Dedekind * drew the inspiration which led him to the theory of the 'cut' in the system of rational numbers, a theory which is fundamental in the Calculus; and further, because the propositions of the Fifth Book furnish a number of easily understood examples of the 'cut.'

To avoid being misunderstood, it is necessary to say that the Seventh Definition and Props. 8, 10 and 13 are valuable in the discussion of *Unequal* Ratios. The point here made is that they are not required for the discussion of *Equal* Ratios.

21. I have now disclosed alike the reason for the difficulty of Euclid's argument and the superstition to which I referred. De Morgan was a logician in the first place, and a mathematician in the second. The logic of the book, except in the matters to which Saccheri and Simson had called attention, was, as De Morgan said, faultless, but he accepted the 5th and 7th definitions on the authority of Euclid, without questioning the necessity of each in relation to the purposes of the book.

I pass now to the propositions. Of these I have said a good deal incidentally, but there are a few things left over.

THE FIRST GROUP OF PROPOSITIONS. Nos. 1, 2, 3, 5 and 6.

22. These propositions deal with magnitudes and their multiples, not with their ratios. With this group I have associated the proposition $r(sA) = s(rA)$.

* Dedekind, *Essays on Number*, translated by Beman, p. 40.

It can be deduced from the 1st proposition by supposing that each of the magnitudes A, B, C, ... is equal to A, and that there are s of them.

Then $r(A+B+C+\ldots)$ becomes $r(sA)$, and $rA+rB+rC+\ldots$ becomes $s(rA)$.

$$\therefore\ r(sA)=s(rA).$$

THE SECOND GROUP OF PROPOSITIONS. Nos. 4, 7, 11, 12, 15 and 17.

23. These propositions are demonstrated by Euclid directly from the Fifth Definition. With them I have associated the 5th, 6th and 7th Propositions of the Tenth Book. It may be noted that Euc. X. 5 can be proved directly by the aid of the 5th Definition of the Fifth Book; that the proof of Euc. X. 6 involves the assumption that any magnitude can be divided into any number of equal parts (see above, Art. 9 (4)); and that Euc. X. 7 is one of the propositions which is relied on by those who argue that Euclid did not regard the ratio of incommensurable magnitudes as a number.

THE THIRD GROUP OF PROPOSITIONS. Nos. 8, 10 and 13.

24. Almost the whole of the proof of Prop. 8 consists in proving that if $A > B$, and if C be a magnitude of the same kind as A and B, then integers n, t exist, such that $nA > tC > nB$.

Since $A > B$, therefore $A - B$ is a magnitude of the same kind as C, and therefore, by Archimedes' Axiom, an integer n exists, such that $n(A-B) > C$.

$$\therefore\ nA > nB+C.$$

Replacing nA by X, nB by Y, and C by Z, this becomes $X > Y+Z$.

It only remains to prove that an integer t exists, such that $X > tZ > Y$.

To do this, let tZ be the greatest multiple of Z next below X, so that either (1) $(t+1)Z > X > tZ$ or (2) $(t+1)Z = X > tZ$.

Since $(t+1)Z \geqq X$ and $X > Y+Z$,

$$\therefore\ (t+1)Z > Y+Z; \qquad \therefore\ tZ > Y; \qquad \therefore\ X > tZ > Y.$$

Replacing X, Y, Z by their values, it follows that $nA > tC > nB$.

(Up to this point ratios are not involved.) The result obtained is the one which is required for the proofs of Props. 16, 22, and 23, which I give in my *Euclid V. and VI.*

With regard to Prop. 10, I have already noted (Art. 12 (*e*)) that the proof in the text assumes that a ratio is a magnitude.

Simson suggests that the proof is not Euclid's, and that it should have been as follows:

If $A : C > B : C$, then by Def. 7 integers r, s exist, such that $rA > sC$, but $rB \not> sC$.

From these it follows that $rA > rB$.

$$\therefore\ A > B.$$

Hence if $A : C > B : C$, then $A > B$.

This proof is not open to criticism.

THE FOURTH GROUP OF PROPOSITIONS. Nos. 9, 14, 16 and 18-25.

25. The propositions in this group relate to properties of *Equal* Ratios, but Euclid's demonstrations depend on properties of *Unequal* Ratios. A very important point is raised by the form of Euclid's proof of Prop. 18. He assumes that if A and B be two magnitudes *of the same kind*, and C be another magnitude, then a magnitude D *of the same kind* as C exists, such that $A : B = C : D$.

This, as was first pointed out by Saccheri, does not follow from Euclid's definitions. Simson showed how the proof could be made to depend on the definitions,* but the proposition assumed is one of great importance. De

*See also my *Euclid V. and VI.* 2nd edition, p. 113, where a proof substantially the same as Simson's is given.

Morgan's attempt to prove it (*l.c.* pp. 60-61 and Heath, Vol. II. p. 171) is insufficient, and in fact the theorem cannot be proved, except in the special case where A, B, C are segments of straight lines, or in cases easily reducible to this case. The result depends on the Axiom by which continuity is ascribed to the system of magnitudes *of the same kind* as C. This Axiom corresponds to the Cantor-Dedekind Axiom, by which the gaps in the system of rational numbers are filled up by the creation of irrational numbers (see my *Euclid V. and VI.* 2nd edition, pp. 162-164).

26. The 22nd and 23rd Propositions bear on what Euclid calls the Compounding of Ratios, but what we should call the Multiplication of Ratios.

Prop. 23 shows that the order of multiplication has no effect on the result.

It is proved that if $A : B = U : V$, and if $B : C = T : U$, then $A : C = T : V$.

Now, if A and B were commensurable, it would be possible to write $A = \lambda B$, and then λ would be a rational fraction. But if A and B are *not* commensurable, *and if we still put* $A = \lambda B$, *we must regard* λ *as the symbol of some kind of operation performed on* B, *which gives as its result the magnitude* A.

Let us further regard the relation $A : B = U : V$ *as justifying the statement that since* $A = \lambda B$, $\therefore\ U = \lambda V$; *i.e.* we consider that U can be obtained by performing on V an operation which, if performed on B, would give A.

In like manner, if $B : C = T : U$, and if $B = \mu C$, then $T = \mu U$, where μ is the symbol of some other operation.

$$\therefore\ A = \lambda B, \text{ and } B = \mu C\ ;\quad \therefore\ A = \lambda(\mu C).$$

$$\therefore\ T = \mu U, \text{ and } U = \lambda V\ ;\quad \therefore\ T = \mu(\lambda V).$$

But by Prop. 23, if $A : B = U : V$, and if $B : C = T : U$, then $A : C = T : V$.

If therefore we put $A = \nu C$, then we must have $T = \nu V$, where ν is the symbol of another operation.

Equating the two values of A, we have

$$\lambda(\mu C) = \nu C\ ;$$

$$\therefore\ \lambda(\mu \text{ and } \nu \text{ are equivalent operations.}$$

Equating the two values of T, we have

$$\mu(\lambda V) = \nu V\ ;$$

$$\therefore\ \mu(\lambda \text{ and } \nu \text{ are equivalent operations.}$$

Hence the operation denoted by ν is the same as those denoted by $\lambda(\mu$ and $\mu(\lambda$ respectively;

$$\therefore\ \lambda(\mu \text{ and } \mu(\lambda \text{ are equivalent operations.}$$

Hence the same result is obtained if the operation μ is first performed and then the operation λ, as is obtained when the operation λ is first performed and then the operation μ.

27. The 24th Proposition corresponds to what we should call the Addition of Ratios. If we attempt to make use of it, or of the 22nd or 23rd Proposition, the question of the existence of the fourth proportional to three magnitudes, of which the first and second are of the same kind, is immediately raised, a matter to which I have already alluded in Art. 25.

28. The 25th Proposition—"If four magnitudes of the same kind are in proportion, then the sum of the greatest and least exceeds the sum of the other two"—is an important one.

My interest in it was first aroused by the fact that De Morgan (*l.c.* p. 72) points out its connexion with the notions on which the first theory of logarithms was founded.

Euclid does not make use of this proposition in his *Elements*. Sir T. L. Heath tells me that it appears once only in Greek Geometry, viz., in the 69th Theorem of the 7th Book of the *Collectiones Mathematicae* of Pappus, where it is used to show that the shortest intercept on all straight lines drawn through the middle point of the base of an isosceles triangle is the

base itself, and that as the straight line turns round the middle point of the base, the intercept made by the sides continually increases until the line becomes parallel to one of the equal sides of the triangle.

It is difficult, if not impossible, to ascertain for what purpose Euclid recorded it. I imagine that most people who look at it think it a comparatively useless result, except in the special case where the second and third terms of the proportion are equal, when it expresses the fact that the arithmetic mean of two magnitudes exceeds their geometric mean.

This special case is, however, very important. It can be used to prove two limits of the utmost value in the Calculus.

(i) If n be a positive integer, and $a>1$, then $\lim_{n\to\infty} a^n=+\infty$.

To see this, let V, V_1, $V_2 \ldots V_n$ be in continued proportion;

$$\therefore\ V : V_1 = V_1 : V_2 = \ldots = V_{n-2} : V_{n-1} = V_{n-1} : V_n.$$

Suppose $V<V_1$, then from $V : V_1 = V_1 : V_2$, it follows by Euc. V. 25 that $V+V_2>2V_1$; $\therefore\ V_2 - V_1 > V_1 - V$.

Similarly, $V_3 - V_2 > V_2 - V_1 > V_1 - V$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$V_n - V_{n-1} > V_{n-1} - V_{n-2} > V_1 - V.$$

From these it follows that $V_n - V_1 > (n-1)(V_1 - V)$; and

$$\therefore\ V_n - V > n(V_1 - V) \quad \text{and} \quad \therefore\ V_n > n(V_1 - V);$$

$$\therefore\ \frac{V_n}{V} > n\left(\frac{V_1}{V} - 1\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since $V<V_1$, if we put $\frac{V_1}{V}=a$, then $a>1$; and from the continued proportion it follows that $\frac{V_n}{V}=a^n$; $\therefore$ (1) gives $a^n > n(a-1)$.

From this, by the aid of Archimedes' Axiom, it follows that, if $a>1$, and n be a positive integer, $\lim_{n\to\infty} a^n=+\infty$. (Cf. De Morgan, *l.c.* p. 72.)

(ii) If n be a positive integer, and $a<1$, then $\lim_{n\to\infty} a^n=+0$.

If in the same continued proportion as in the last case we put $\frac{V}{V_1}=a$, then $a<1$, and $\frac{V_1}{V_n}=a^{n-1}$.

Now, from $V_n > n(V_1 - V)$ it follows that $\frac{V_n}{V_1} > n\left(1-\frac{V}{V_1}\right)$,

$$i.e.\ \frac{1}{a^{n-1}} > n(1-a);$$

$$\therefore\ \frac{1}{n(1-a)} > a^{n-1} > a^n; \qquad \because\ a<1, \quad \therefore\ \frac{1}{n(1-a)} > a^n.$$

From this, by the aid of Archimedes' Axiom, it follows that $\lim_{n\to\infty} a^n=+0$, where $a<1$, and n is a positive integer.

Upon this last result depends the proof of the convergency of an infinite geometric progression, when the common ratio is less than unity.

Euclid gives a proof of a result which includes the theorem that $\lim_{n\to\infty} a^n=+0$, if $a<\frac{1}{2}$ in the first proposition of the Tenth Book, which he uses in the Twelfth Book in his determination of the areas of circles and the volumes of tetrahedra.

One may perhaps be tempted to believe that he had by means of Euc. V. 25 obtained a proof of the result $\lim_{n\to\infty} a^n=+0$, not merely for the range $0<a<\frac{1}{2}$, but also for the wider range $0<a<1$; but that he put it aside

because he was able to express the demonstration more simply for the narrower range, as he did in Euc. X. 1, than for the wider one ; and because the narrower range was sufficient for the geometrical applications. But this of course is mere speculation.

THE FUNDAMENTAL CRITICISMS OF THE FIFTH BOOK OF EUCLID'S ELEMENTS.

29. Let me in taking leave of the book sum up the fundamental criticisms on the text as it has come down to us.

(i) There is the point first noticed by Saccheri that Euclid in his demonstration of Prop. 18 assumed the existence of the fourth proportional to three magnitudes, of which the first and second are of the same kind.

Simson supplied a proof of Prop. 18 which is not open to criticism (see Art. 25).

(ii) There is the point noticed by Simson that Euclid in his proof of Prop. 10 assumes more about ratios than his definitions allow him to do.

Simson supplied a proof of Prop. 10 not open to criticism (see Art. 24).

(iii) There is the point noticed by Stolz that one of the three sets of conditions contained in the Fifth Definition is involved in the other two (see Art. 14).

(iv) There is the point, first published in 1897, which forms the main subject of this address, viz. : that the properties of *Equal* Ratios are all deducible from the Fifth Definition, and that it is not necessary to employ the Seventh Definition, which is concerned with *Unequal* Ratios, for this purpose.

When account is taken of these matters, the artificial character of the proofs, and the great difficulties resulting therefrom, all disappear.

CONCLUSION.

30. And here I bring my address to a close. I have endeavoured to tell you something of the contents of this great work, something of what it did not contain, but which it directly suggests ; and I have tried to show you how near the line of argument runs to the ideas now accepted as giving an adequate account of the theory of irrational number.

I hope that I have at the same time succeeded in communicating to you some small portion of the affection and reverence I feel for this great masterpiece, perhaps the noblest monument of the Greek intellect.

It is a great treasure house of ideas. There are two doors to it. One, representing the Euclidean text, stands locked, bolted, and barred, and over it is written, "Let none but the accomplished mathematician strive to enter here." The other, representing the arrangement of the argument explained above, stands wide open for every one who has a knowledge of Elementary Algebra. If, as I hope, I have shown you that no one so equipped need despair of a welcome in this house, I shall be well content.

ON THE TEACHING OF MATHEMATICS IN ELEMENTARY SCHOOLS, AND ITS CONTINUITY WITH THE TEACHING IN SECONDARY SCHOOLS.*

ARITHMETIC has been until recently the only branch of Mathematics generally taught in the Primary School ; and as this subject deals exclusively with numbers in their simplest and most practical form, it must, in the order of teaching, be the first introduced to the child. The premier place must also be awarded to it when considering the relative importance of the various branches composing the unity of Mathematics.

* A paper read before the N. Wales Branch.

During the last decade, however, Algebra and Geometry and even Trigonometry have been introduced into the Primary School by several teachers, and it is gratifying to find that the experiment has met with unqualified success. But how far mathematical unity can be pursued in the teaching of the Primary School is a matter of considerable debate, and a question that would be better left to the individual teacher to decide.

In the teaching of Mathematics there is a point beyond which the Primary School cannot go, just as there is a division between the Secondary School and the University. But where exactly that point is, it is a very difficult matter to determine. It varies according to the locality and the conditions under which a school labours. And yet to limit the extent of the work would rob the children of a part of their heritage, and quench the enthusiasm and individuality of the teacher.

Less than twenty years ago the work of the Primary School was set out in detail by the Education Department (as it was then called), and as an example of the style of work then asked for, I will read to you the following:

First Year (children of 7 and 8 years of age): Notation and numeration up to 1000. Simple addition and subtraction of numbers of not more than three figures. In addition not more than five lines to be given. The multiplication table to 6 times 12.

Mental Arithmetic: Short exercises, not involving large numbers and dealing with concrete as well as abstract numbers, preparatory to the work of the next standard.

Second Year (children of 8 and 9 years of age): Notation and numeration up to 100,000. The four simple rules to short division. The multiplication table and the pence table to 12s.

Mental Arithmetic: Same wording as above.

Third Year (children of 9 and 10 years of age): The former rules with long division. Addition and subtraction of money.

Mental Arithmetic: As above.

Fourth Year (children of 10 and 11 years of age): Compound rules (money) and reduction of common weights and measures, viz. avoirdupois weight, long measure, liquid measure, time measure, square and cubical measure.

Mental Arithmetic: As above.

Fifth Year (children of 11 and 12 years of age): Practice, bills of parcels, and simple rule of three by the method of unity. Addition and subtraction of proper fractions with denominators not exceeding 12.

Mental Arithmetic: As above.

Sixth Year (children of 12 and 13 years of age): Fractions, vulgar and decimal. Simple proportion and simple interest.

Mental Arithmetic: As above.

Seventh Year (children of 13 and 14 years of age): Compound proportion, averages and percentages.

You will observe that almost all the teaching was concentrated upon Arithmetic alone, as if it were self-contained, with no relation to geometry and the physical sciences. The time expended upon working long, cumbrous, unnecessary examples was sheer waste. There was no practical or moral gain to the pupil, who might have been far more profitably employed by taking up another branch of the subject.

The great defects of the syllabus were:

1. It was based on the Text-Book and not on the development of the child-mind.
2. It ignored the knowledge acquired by the child out of school.

With the advent of the new century, the Board of Education, in publishing their typical schemes, granted to teachers the liberty to adopt them in their

entirety or to frame their own, subject to the approval of His Majesty's Inspectors. This was a concession heartily welcomed by the teachers, and although it imposed upon them more work and greater responsibility, it stimulated many to put into operation ideas that had been simmering in their minds for some time past. As a result, many original and intelligent schemes have been evolved, which, as far as can be judged at present, have proved eminently successful.

In the Infants' School practically no formal Arithmetic is taken. The little children have acquired first-hand knowledge of a pint of milk, a pound of butter, a yard of ribbon. Who knows better than a child at this age the buying capacity of a penny? The coalman with his cart and bell has a curious fascination for the child. Is it impossible that he has no idea of a cwt. of coal? That young children have some knowledge of the rudiments of geometry may be seen from their correct use of such words as: top, bottom, above, below, big, little, as far as, about the middle. These facts and experiences should be utilised at the earliest possible stage in the child's school life, and made the basis of the teaching in the science of number. If these experiences were noted and collated and allowed to crystallise, they would be of inestimable value in giving young children an all-round training, and would thus prepare the way for more advanced work in after years. It is true that this is not the method of the text-book, which as a rule commences with notation and numeration, and then proceeds to deal with each rule in succession. There is no book that can record on one page, or even in one chapter, the wonderful and varied activities of the human mind. We have also learned that there is a danger of striving after pedantic thoroughness by insisting upon a complete and accurate knowledge within a small compass of numbers. And this undoubtedly the "Committee of Fifteen" had in their minds when they reported some years ago: "The attempt to secure what is called thoroughness in the branches taught in the elementary schools is often carried too far; in fact, to such an extent as to produce arrested development in the mechanical and formal stages of growth. The mind in that case loses its appetite for higher methods and wider generalisations." Very true! The outlook at the beginning of his career was so dwarfed that the child became stunted in his mental activities. The background for young children should be as large as possible, yea, without boundaries—the real and the imaginary blending together in the most subtle and seductive manner. This I take to be the dominant characteristic of modern teaching as applied to the Primary Schools.

Generally speaking, the course now followed is somewhat as follows:

Instead of spending a whole year on addition and subtraction involving the use of hundreds, the teaching divides itself into four sections:

1. Practical and Experimental. 2. Mental and Oral. 3. Written. 4. Manual Work.

These divisions run concurrently throughout the years during which the child is in school.

Standard I. Inch cubes and laths are used for building up simple numbers involving no more than the tens; also dummy coins, ruler and tape, simple weights and measures (lbs. and oz., pints and quarts); the clock. No number to exceed 60.

Standard II. Subjects of Standard I. extended. Simple fractions; the decimal system with ruler for inches and tenths. Numbers not to exceed 99.

Standard III. Fractions, decimals; metric system (metre and multiples, etc., only); mensuration of simple areas. Algebra, use of symbols in the four rules.

Standard IV. Subjects of Standard III. extended; easy proportion, plotting graphs for attendance, thermometer, etc. Algebra, use and meaning of $a+b$, $a-b$, $a\times b$, $\frac{a}{b}$; easy simple equations and problems.

Standard V. Harder exercises in I.–IV. Metric system, in which long names of multiples and submultiples are needless—they can be expressed decimally. Fractions in algebra.

Standard VI. All previous work thoroughly revised and somewhat extended. Percentages, averages, interest, discount. Graphs, charts, diagrams. Algebra, factors, simple fractional equations, simultaneous equations.

Standard VII. Simple surds; A. and G. progressions; easy indices and logarithms; with a full course in Arithmetic.

Each section mentioned above opens out a well-nigh inexhaustible store of exercises and sub-exercises which only an interested and intelligent teacher can manipulate. And herein lies the secret of successful teaching of number to young children and the superlative need for careful thought and preparation on the part of the teacher to come before his class armed with all those weapons so essential to create interest in the work, and to draw the children almost unconsciously into the vortex of intensive mental effort.

Geometry might be begun much earlier, and theoretical geometry introduced.

Although this rough outline of what is being done is more or less applicable to all schools, especially in the earlier years, the work in the later years is very much curtailed and handicapped by the miserable requirements of some educational authorities for scholarships to be held in the Secondary Schools. Until this anomaly is removed, it will be utterly impossible for all schools to adopt a rational and an intelligent scheme similar to that I have just described.

Not till the fifth year do some children pass on to the County Schools with a fair knowledge of Arithmetic, Algebra, elements of Geometry and Mensuration.

Glanadda Council School, Bangor. R. W. Jones.

ON THE TEACHING OF ELEMENTARY MATHEMATICS IN SECONDARY SCHOOLS.*

In order to make this brief paper as practical as possible, I shall plunge into the subject with little introduction.

I wish to consider the question from three points of view.

(1) What is the ideal course?

(2) What is the best practicable course?

(3) What effect the best practicable course should have on the teaching of the subject in our secondary schools, and also—since the two are so closely connected—in our elementary schools.

Of course, the chief difference between the ideal and the practicable largely arises from the fact that we are not able as teachers to frame our syllabus exactly as we wish; external forces—chiefly our British external examination system—make us tone down our ideal into the practicable, but I hope to show that even with this limitation we can still do much to make our elementary mathematical teaching more efficient than it has been. And in what follows I wish it to be understood that I am thinking of the average child, though I may say that it is too often assumed that the elementary

* A paper read before the N. Wales Branch.

mathematical course suited to the average child is unsuitable for the budding mathematician. I fail to see why this should be so.

But to return to the main topic and to consider the subjects we have to deal with seriatim.

Arithmetic.—We must teach our pupils both to think and to calculate, and to use their powers of calculation in connection with problems more or less close to their experiences of daily life. We now allow them to shorten considerably their solutions by using algebraical methods in reason, though, while considering such algebraic solutions as correct, I myself almost invariably show my class for the benefit of the abler boys purely arithmetical solutions. "Do them by arithmetic, if you can, but do them." In addition to the fact that x and y are not now tabooed in arithmetic, another great change (of which we have not yet seen the full development) is the use and application of approximate methods of calculation. But for such methods to be effective they must be made "part and parcel" of the whole arithmetic course. They ought to be developed so that instead of asking for answers to a certain number of significant figures or decimal places, we should be able to ask later on for answers to as many figures as are permitted by the data, considering the latter to be subject to error. An article in the *School World* for last December by the Mathematical Master at Manchester Grammar School indicates one way of tackling such problems. And may I say in an aside at the expense of our friends who teach Natural Science that such a method is more scientific than the "Always work to four figures," which is so frequently used.

To what extent does the pressure of necessity (I mean our sacro-sanct external examination system—with most of us the Central Welsh Board) affect our arithmetic syllabus and the teaching of it? I believe it need not affect the sort of work I have referred to above. I know of no arithmetic examination in the country where the pursuing of rational methods of approximation meets with its just reward to the same extent as it does in the C.W.B.—in the Junior, though of course especially in the Senior.

I think a teacher who prefers what may be called rectangular solutions to pyramidal solutions of the old type need not fear to submit his pupils to the test of an examination like this. Of course, if we had no external examination we might be able to spend rather more time on interesting though out-of-the-way problems, but if any British examination in arithmetic known to me deserves well of modern teachers it is the Central Welsh Board.

I have dealt with the subject of approximation at full length. To what extent does it affect our elementary work and the relationship 'etween that work and the teaching in the elementary schools from which our pupils largely come? It goes without the saying that the primal necessity is for us to teach our elementary rules, multiplication of course especially, in such a way as to help and not hinder our approximate work later on. This refers to the need of multiplying by the left hand figure of the multiplier first of all. At present the other method becomes so ingrained in the pupil's mind that there is always a chance that in the question: Multiply 16·81 by 20·575, the child uses the one method of starting, while if the question reads: Multiply 16·81 by 20·575 (correct to the nearest whole number) the other method is (reluctantly) employed. And since even the youngest children are taught multiplication before they enter our secondary schools, I should like to pass on this recommendation to our friends who teach in elementary schools, that multiplication should begin with the left-hand digit of the multiplier, and that this should be done from the first if possible, but at all events in the higher standards.

There are other matters which should claim our attention in arithmetic, but I have no time to do more than refer to a few of them by name.

Decimals and vulgar fractions should be begun almost together, decimals

if anything having the preference as regards order, and the metric system provides an excellent concrete basis. Recurring decimals have, I believe, gone from our midst altogether, with the exception perhaps of the notation, not to appear again until convergent geometrical progression is dealt with.

Algebra.—In connection with a modern syllabus in Algebra I cannot do better than refer you to Mr. C. Godfrey's pamphlet on *The Algebra Syllabus in the Secondary School*, which was published last year by the Board of Education. He draws up a syllabus for non-specialist pupils, *i.e.* for children who need mathematics only as a part of their general education, and he considers that the aims we should have should be—

(1) the solving of problems;

(2) the study of functionality, including variation, graphs and the like.

This would give us time to introduce to our ordinary forms numerical Trigonometry and the ideas of the infinitesimal calculus, a view not so startling if we accept Prof. Love's dictum that (I quote from memory) "The principles of the calculus ought to be counted as a part of the intellectual heritage of every educated man and woman in the twentieth century, no less than the Copernican system or the Darwinian theory." Of course, the question of the Calculus hardly concerns us here, unless we treat it purely as a branch of Elementary Mathematics, a by no means impossible thing to do, but we can still consider the advisability of leaving out much of the algebra we at present deal with in order to make room for numerical Trigonometry and the Calculus. But we who are influenced by an external examination can hardly alter our syllabus to the extent indicated during this year or next year at any rate. What then can we do to make our algebra more concrete and valuable in view of our limitations? I still think that we may use the two aims I have taken from Mr. Godfrey's pamphlet as a sort of standard-ideal, if you will, though the word suggests its present impracticability. If we are trying to teach the way to solve problems, and if we hope later on to add the study of functionality (always remembering the examination), we can do much to make our algebra more reasonably modern and useful without rendering it any the less logical.

Consider the following syllabus in connection with our Junior stage:

Simple questions on notation and symbolical expression.
Easy simple equations and problems.
Addition, subtraction, multiplication and division of monomials.
"Think of a number" problems.
Harder simple equations and problems.
Linear equations in two unknowns and problems.
Linear statistical graphs in their many applications.

Now can we alter in this way the order of our algebra work with our present limitations? I think we can, though at present there may be some little element of risk. I think we have a right to ask that this risk be eliminated. Surely there is no need for such an order as "substitution, the four rules, simple equations, factors, H.C.F. and L.C.M., fractions" to be perpetuated in both the examination paper and in the syllabus, unless we are expected to consider this the best order in which we can teach the syllabus. Is it too much of a revolution to ask for a couple of equations to be solved in the first question, and then for one or two problems to follow? Our pupils are not good at problems, but I ask whether under our present method and order they are likely to be anything else.

With regard to the relation of our elementary algebra work to the algebra in the elementary school, I think it is fairly obvious that (where algebra is taught at all) it should be different from the five-worded syllabus "substitution, addition, subtraction, multiplication, division." The early part of the scheme sketched out above can be followed—you have not to consider the pressure of external examiners. Simple equations and problems can be

dealt with as soon as $3x$ can be doubled, the questions, of course, being carefully chosen. As soon as the negative sign is understood and addition and subtraction of monomials have been attended to, slightly harder equations and problems can be taken. The "think of a number" problems can be solved and invented by quite small children (I tested it during a summer holiday with a girl of ten years who had never learnt a word of algebra). The value of the exercise is not reduced because of the pleasure the inventor shows when he can say in his oracular manner "Every one of you should have got the answer 23," or "Your answer, John, is 19; then the number you thought of was 7."

Finally, if multiplication is taught beyond that of monomials, surely the horizontal multiplication of simple expressions (especially binomials) should be taught some time before the ordinary multiplication in columns is introduced. Only by so doing can we be spared the irritation of seeing a boy work out two-thirds of a multiplication sum in the margin when he has to multiply $2a+3b$ by $5a-2b$.

Geometry.—The superiority of the new geometry over the old Euclid is incalculable, though the tendency to looseness of thought in connection with proofs is a danger to be carefully guarded against. "Intuition" is an excellent aid to enabling a child to see the truth of a theorem, especially if carefully drawn figures are not used. (A few pupils will say, "These lines are equal, just notice their lengths," and so I am a firm believer in freehand figures, or at any rate figures not drawn to scale, being used in connection with propositions.) But surely intuition need not be always invoked in order to avoid proving a proposition. In the early work a simple proof of an axiomatic proposition like Euc. I. 13 (the Theorem I. of nearly every text-book) is not without its value.

What should be the relation between our practical and our theoretical work? My own view is that after a very short course in practical geometry, not lasting longer than a term at most, the theoretical work should be begun, and the two henceforward carried on side by side. In the early stages practical work will generally enable a child to discern the truth of a new proposition, and as a rule a class can be made to teach the teacher a proof of it—of course, with a little judicious help.

We should ever be trying to give our pupils the power to solve riders. To me a course of propositions in geometry is something akin to drudgery; but the riders are quite as precious as are the pictures to a boy of fourteen as he reads his book of adventure. How can we make them so to our pupils? And perhaps in this connection one might be allowed to refer to Prof. Bryan's plea in the *School World* for a fixed order of propositions. I am wholly with him in that plea. I could *almost* agree with him when he says, "Euclid's order is better than no recognised order at all." Who has not been irritated by having I. 5 proved by I. 26 instead of by I. 4, a thing that happens largely because of the general vagueness on the question of order, especially if a boy happens to change his text-book when he changes his school? Surely it is not impossible to hope for a representative committee which shall decide upon a *numbering* of propositions. The order in which such propositions would be learnt could be varied, and this took place even in the days of the iron rule of Euclid. The freedom of the past few years has taught us much in the way of what to avoid.

Trigonometry.—The usefulness of Trigonometry in the way of practical applications is probably greater than that of either algebra or geometry, and therefore it is a pity it is not studied more than it is. And yet one first of all needs a certain groundwork of both of these other subjects. "Pythagoras" needs to be known at any rate, for Trigonometry, which only depends upon drawing to scale, will not carry us far. And if a pupil's algebra is so vague that the factors of a^2-b^2 are ever in doubt, I hardly see

why Trigonometry should be added to his repertoire of knowledge—or ignorance. But a child can learn much of useful Trigonometry in a very short time, and as a first course the solution of triangles without the use of the addition formulae may well be aimed at. Drawing to scale should be used—in moderation, and with pupils of the Junior Form much may safely be attempted—or might be, if only the time could be found for it in the syllabus. An hour and a half a week would be very effective as a means of introducing to young pupils the elements of numerical Trigonometry, including logarithms. It is in this subject, I think, that logarithms ought to be taught rather than in algebra, though if Trigonometry is not taught at all they ought to be included in the algebra course for the sake of their practical usefulness. The drawing of the graph $y=2^x$ and its interpretation as the graph of $x=\log_2 y$ forms an excellent introduction, 2 being a better base than 10 for illustrative purposes, because 2^n increases with n less rapidly than 10^n.

Wrexham County School. W. J. WALKER.

MATHEMATICAL NOTES.

379. [L¹. 1. c.] *Pascal's Theorem; Brianchon's Theorem; Cross-Centre and Cross-Axis.*

(i) *Pascal's Theorem.*

Let A, B, C, D, E, F be six concyclic points. Let AB, BC, CD cut DE, EF, FA in L, M, N. Then L, M, N are collinear.

Let the circumcircle of MBE meet DE, AB in H, K.

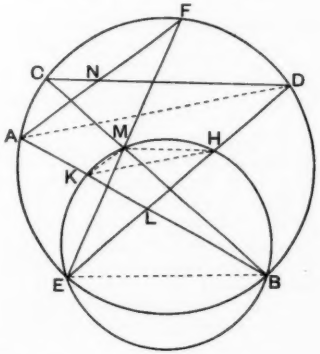
Then $\angle MHE=\angle MBE=\angle CDE.$

$\therefore MH \parallel CD.$

Similarly $HK \parallel DA$ and $KM \parallel AF,$

i.e. sides of $\triangle MHK \parallel$ sides of $\triangle AND.$

$\therefore NM$ passes through $L.$



(ii) *Brianchon's Theorem.*

Let the sides AB, BC, CD, DE, EF, FA touch the circle in $P, Q, R, S, T, U.$

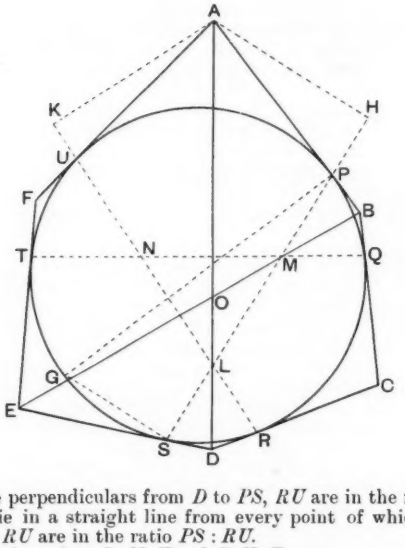
Draw AH, AK perpendicular to $PS, RU,$ and let PG be the diameter through $P.$

Then, since the sides of $\triangle AHP$ are perpendicular to the sides of $\triangle PSG$,

$$\therefore \frac{AH}{AP} = \frac{PS}{\text{Diameter}}.$$

Similarly
$$\frac{AK}{AU} = \frac{RU}{\text{Diameter}},$$

$$\therefore \frac{AH}{AK} = \frac{PS}{RU}.$$



Similarly the perpendiculars from D to PS, RU are in the ratio $PS : RU$.

$\therefore$ A, L, D lie in a straight line from every point of which the perpendiculars to PS, RU are in the ratio $PS : RU$.

Similarly for the points B, M, E and C, N, F.

Hence, if AD, BE intersect in O, the perpendiculars from O to RU, QT are in the ratio $RU : QT$.

$\therefore$ O lies on the straight line CNF.

By a similar demonstration it follows that: (i) If from a given external point tangents be drawn to a given circle, the perpendicular from it to a chord through either point of contact bears a fixed ratio to the chord.

(ii) If the sides AB, BC, CD, DA of a quadrilateral touch a circle in P, Q, R, S, then the diagonals AC, BD and the chords PR, QS form a harmonic pencil, for the perpendiculars from each of the points A, B, C, D to PR, QS are in the ratio $PR : QS$.

The property used to demonstrate Brianchon's Theorem may be used independently to establish the existence of the Cross-Centre and of the Cross-Axis.

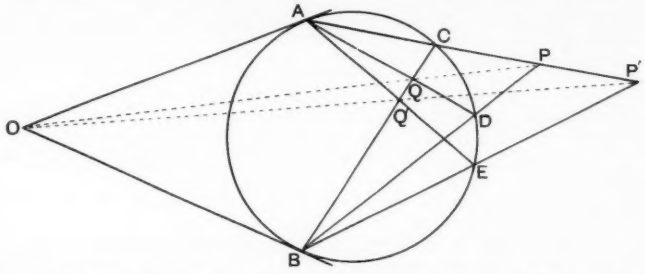
(iii) *The Cross-Centre.*

Let the points A, B, C, D lie on the circle $ABDC$, and let AC, BD cut in P; AD, BC in Q; then PQ passes through the pole O of AB.

For the perpendiculars from O to BC, AD are, as above, in the ratio $BC : AD$.

Also, by similar triangles PBC, PAD, the perpendiculars from P to BC, AD are in the ratio $BC : AD$.

$\therefore$ O and P lie on a straight line through Q, *i.e.* PQ passes through O.



Similarly, if any fifth point E be taken on the circle and AE, BE cut BC, AC in Q', P' respectively, then $P'Q'$ will pass through O.

The property follows by projection for any conic.

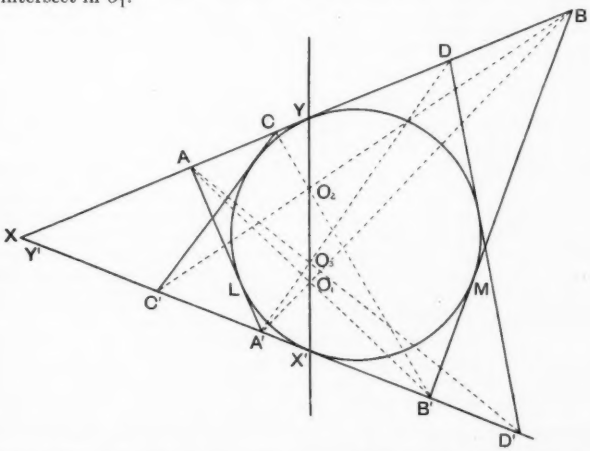
Hence, if five points A, B, C, D, E are given on a conic, any number of others may be found.

For (AD, BC), (AC, BD) determine PQ, and (AE, BC), (AC, BE) determine $P'Q'$.

Hence the cross-centre O may be found; and, taking OP'' arbitrarily to cut AC, BC in P'', Q'' respectively, BP'', AQ'' determine a point F on the conic.

(iv) *The Cross-Axis.*

Let the sides AB, BB', $B'A'$, $A'A$ of the quadrilateral $ABB'A'$ touch a circle in Y, M, X', L respectively, and let LM, $X'Y$ cut in O_1; then AB', $A'B$ intersect in O_1.



For the perpendiculars from A to LM, $X'Y$ are in the ratio $LM : X'Y$, and also the perpendiculars from B' to LM, $X'Y$ are in the ratio $LM : X'Y$.

$\therefore$ A, B' lie on a straight line through O_1.

Similarly A', B lie on a straight line through O_1.

$\therefore$ AB', $A'B$ intersect on $X'Y$.

Similarly, if any fifth tangent CC' cut the given fixed tangents XAB, $XA'B'$ in C, C', then will BC', $B'C$ intersect at a point O_2 on $X'Y$.

The property follows by projection for any conic.

Hence, if five tangents to a conic are given, any number of others may be drawn.

For AB', $A'B$ determine O_1, and BC', $B'C$ determine O_2.

Hence the cross-axis $X'Y$ may be found; and, taking D arbitrarily on XAB, $A'D$ cuts the axis in O_3, and AO_3 determines D' on $XA'B'$.

E. M. LANGLEY.

380. [I. 18. c.] *Note* 357. *Powers of numbers whose sum is the same power of some number.*

On p. 197 of the *Gazette* (No. 95, Dec. 1911), Dr. Barbette says: "I must add that until the appearance of the note in the *Gazette* (No. 94, Oct. 1911, pp. 158-9) I was totally unaware that Mr. Artemas Martin had ever been engaged in researches analogous to my own." The facts in the case are that he had in his possession my papers on fourth and fifth power numbers some months before the appearance of my note to which he refers.

I wish to add here a couple of early references to the publication of fourth powers whose sum is a fourth power, and of fifth powers whose sum is a fifth power, overlooked when preparing my former note.

Dr. David S. Hart proposed in the *Educational Times* as Quest. 3115, "To find five biquadrate numbers whose sum is a biquadrate number."

A solution by the writer, and Rev. U. Jesse Knisely, finding the numbers

$$4^4+6^4+8^4+9^4+14^4=15^4,$$

was given in the *Reprint*, Vol. XX. (London, 1874), p. 55, 36 years before the publication of Dr. Barbette's monograph.

I proposed in the *Educational Times* as Quest. 9563, "Find six whole positive numbers the sum of whose fifth powers is a fifth power." My solution, finding the two following sets of such numbers,

$$4^5+5^5+6^5+7^5+9^5+11^5=12^5,$$

$$5^5+10^5+11^5+16^5+19^5+29^5=30^5,$$

was published in the *Reprint*, Vol. L. (London, 1889), pp. 74-5, 21 years before the appearance of Dr. Barbette's monograph.

Washington, D.C., Sept. 13, 1912. ARTEMAS MARTIN.

We have received from Dr. Barbette the following note: "My work on p^{th} powers was completed in Oct. 1909 and published in July 1910. Dr. Martin purchased a copy from me direct, and in Nov. 1910 I received from him a complimentary copy of his paper 'About biquadrate numbers whose sum is a biquadrate.' This does not contain the method of exclusion developed in my volume, nor does it contain the identity

$$4^4+6^4+8^4+9^4+14^4=15^4.$$

It was much later, in 1911, that he sent me his other papers dated 1888, 1893 ..., and these contain not only the identity above, but also

$$4^5+5^5+6^5+7^5+9^5+11^5=12^5.$$

"Quite recently I wrote to Dr Martin acknowledging the priority of such results of his on p^{th} powers as had appeared before the publication of my volume in July, 1910. If I had only known of his researches, my book would have been more important than it is, for I would, with his consent,

have added to it the curious artifices which he has utilised in his researches for the purpose of exhibiting sums of p^{th} powers as a p^{th} power in the cases $p=4$, 5, and even 6."

381. [I. 17.] No. 99, vol. vi., July 1912, p. 338, Note 373. Identity (v) does not hold for $x=1$. What is the correct form? E. N. BARISIEN.

QUERIES.

(83) In an ordinary pack of cards, what is the chance that a King and Queen come together? G. N. B.

(84) What is the origin of the sign $\doteqdot$ denoting "is very nearly equal to"? J. M'WHAN.

REVIEWS.

A B C of Hydrodynamics. By Lieut.-Col. R. DE VILLAMIL, R.E. Pp. 135. 6s. net. 1912. (Spon.)

What is hydrodynamics? According to the classification in the 9th edition of the *Encyclopaedia* it is a purely mathematical investigation of the equations of motion (and their integrals) of a purely hypothetical medium called a perfect fluid (sometimes also a viscous fluid), sometimes enclosed in or bounded by surfaces of various forms, such as infinitely long elliptic cylinders or confocal ellipsoids, sometimes containing sources or doublets such as could not possibly be produced experimentally. It really consists, to all intents and purposes, in a study of certain partial differential equations, and it fails to take account of the conditions prevailing in the practical engineering problems associated with town water supply, resistance of ships, screw propellers and aeroplanes. Still the subject is studied by engineering students, probably to their advantage.

According to this definition, Col. De Villamil's book has nothing whatever to do with hydrodynamics, and its title would perhaps have been correct if the author had described it as the A B C of *Hydraulics*. It deals with such questions as the inapplicability of properties of an infinite liquid to liquids of finite extent, the various forms of fluid resistance, such as those due to viscosity, discontinuous motion, vortex motion and eddy formation, and their dependence on the assumed properties of the liquid. The author examines the general conclusions of such writers as Lord Kelvin, Lord Rayleigh, Stokes, Lamb and other theorists in the light of such experiments as those of Osborne Reynolds, Hele Shaw, Poiseuille, Lanchester and others. His book contains no long abstract equations, but rather statements of experimental evidence. While the author evidently holds strong views on the subject, he is very careful to make the book largely a critical discussion of the views of others. A summary of conclusions is given at the end of each chapter. Some of these conclusions will be readily admitted by most people, for example the statement on page 128 that the resistance experienced by moving bodies is chiefly caused by eddy formation. On the other hand, no useful purpose would be served in this review in entering upon a controversy as to whether the mathematician's ideal, "continuous incompressible inviscid infinite liquid," involves contradictions of terms, thus resembling the irresistible body meeting the impenetrable obstacle. There are not very many inducements to a mathematician to try and adapt his work to the requirements of engineers, and if he really wishes to do so he will generally find it better to become an engineer and give up most of his mathematics, relying on the introduction of *constants or coefficients* to save him from running his head against insoluble differential equations. On the other hand, the hypothetical conditions assumed by the mathematician have pretty well done their duty when they have been made use of to write down differential equations, and it may be argued that the inconsistencies of which Col. De Villamil complains do not apply to the equations themselves.

The real difficulty of the position is indeed remarkably simple. As soon as you give up the equations of a medium derived from the assumed definitions of an ideal fluid, you obtain equations of which integrals cannot be found; at least, mathematicians have tried over and over in vain to find them. The same difficulty

occurs with discontinuous motion in three dimensions, and it is bound to occur in cases other than those of steady motion—for example, eddy formation in the rear of planes. It is no use for Sir William White to urge mathematicians to obtain a hydrodynamical solution of the problem of the screw propeller.

In these days of over-elaboration, the small size of the book is an advantage. It is particularly suited for engineers and engineering students, who will doubtless find the explanations very helpful. G. H. BRYAN.

Engineering Applications of Higher Mathematics. By V. KARAPETOFF. Part I. Problems on Machine Design. Pp. xiv+69. $0·75. 1912. (John Wiley & Sons, New York.)

This book is the first of a series, in which the author intends to discuss typical problems in the various branches of Engineering. Subsequent volumes of the series will deal with Hydraulics, Strength of Materials, Thermodynamics and Electrical Engineering. The problems are all well chosen to illustrate the applications of various branches of Mathematics, and have no connection the one with the other, with the result that the student can choose any chapter that deals with a subject in which he is particularly interested. The first problem dealt with is the Inclined Plane and Screw, and in this the analysis is mainly trigonometrical. The remaining problems deal with Friction in Journals and in Step Bearings, Carrying Capacity of Belts, Torsion of Shafts and Moments of Inertia of Flywheels. These are treated mainly with the aid of the Calculus. Each problem is worked out in detail, and the mathematical analysis shows considerable ingenuity and freshness. Problems, to be worked by the student, are suggested at the end of each chapter on the subject matter which has been discussed. The scheme of the book is admirable, and no student of Engineering can fail to benefit by working through any one of the chapters of which the book is composed.

It would seem from the preface that not so many years ago the teaching of Mathematics to Engineering students in America was at least as much divorced from the practical side as it was in England, and the author pays a generous tribute of praise to Professor Perry and to his *Calculus for Engineers* for his pioneer work in Engineering Education.

Strength of Materials. By H. E. MURDOCK, M.E., C.E. Pp. xiv+296. n.p. 1911. (John Wiley & Sons, New York.)

In this work we have a complete discussion of the subjects generally dealt with under the title of "Strength of Materials." In the chapters on Direct Stresses, Rivetted Joints, Shearing Force and Bending Moment, the usual results and formulae are clearly stated. Diagrams are used extensively, and they are in nearly all cases clear and convincing. Special attention is given to the mathematical connection between Load, Shear and Bending Moment Diagrams, a point which frequently receives but scant notice in text-books. Though no objection can be taken to the introduction of the method of Graphic Integration as giving the student "an additional avenue of conception" (*preface*), and as indicating clearly the meanings to be attached to the various constants of Integration, yet when this method is actually applied to the determination of the slope and deflection curves for Continuous Beams, it becomes so tedious and dull as to defeat its own end. The author should have remained content with applying the method to the simpler cases. The usual way of deducing these curves by integration of the equation $\frac{d^2y}{dx^2}=\frac{M}{EI}$ is given in an admirably short and concise chapter. The book concludes with short chapters on Secondary Stresses, Columns and Struts, Torsion of Shafts, and Resilience, and Hysteresis, which afford good introductions to these branches of the subject. R. M. MILNE.

Examination Papers in Mathematics. By P. A. OPENSHAW. Pp. 135; with answers. Price 1s. 6d. (Bell & Sons.)

This collection of examination papers set for Entrance Scholarships at the Public Schools is chiefly designed for the guidance of Preparatory Schools. It might well serve a wider purpose in providing material for revision exercises in other schools, a large number of the examples being original in type, such as are not usually met with in the generality of text-books.

Examples from a Geometry for Schools. By F. W. Sanderson and G. W. Brewster. Pp. 147; with answers. Price 1s. 6d. (Cambridge University Press.)

These should be extremely useful, especially for those teachers who have a preference for the method of teaching Geometry by practical applications and calculation.

Exercises from The Calculus for Beginners. By J. W. Mercer. Pp. 160; with answers. Price 3s. (Cambridge University Press.)

An excellent set of exercises for use without a text-book, or in conjunction with any text-book. There is a refreshing variety about the set which will appeal to all those who only have at their command the usual stereotyped kind.

Numerical Trigonometry. By J. W. Mercer. Pp. 157; with answers. Price 2s. 6d. (Cambridge University Press.)

Examples in Numerical Trigonometry. By E. A. Price. Pp. 90; with answers. Price 2s. (Cambridge University Press.)

These two volumes are written with the same excellent idea, the inculcation of neatness, orderliness and accuracy in arithmetical work, especially decimals, by a method which will *interest* the pupil. A secondary idea is the practical one of making all boys able to use tables whenever necessary: this idea might well have been carried further than it is; for, although we find certain applications to mechanics, variety and utility might have been combined in giving miscellaneous sets of formulae occurring in Physics and Engineering for evaluation under given data. Naturally the "bookwork" is cut to a minimum, but what explanations are given are satisfactory.

Elementary Mathematics. Part I. By P. V. Seshu Aiyar, B.A., L.T., and V. Venkasubbayya, B.A., L.T. Pp. 489. (Srinivasa Varadachari & Co., Madras.)

This book is written with special reference to Indian conditions, and develops concurrently Arithmetic, Mensuration, Algebra and Geometry. The elementary rules are exceedingly well done, graphical illustration being added from the start; and plenty of examples from every-day life are given. Decimals are introduced by concrete examples from the metric system. Vulgar fractions are deferred, and here perhaps the proof of the rules for multiplication and division by a fraction are not quite as good as the rest of the section. The fundamental laws of Algebra are obtained by generalization, as are also the fundamental theorems of Geometry, though the student is continuously recommended to prove his generalizations by reasoning, according to examples of method given here and there. There is a good section on graphs, special attention being paid to the straight line as illustrating direct proportion. There are a large number of revision papers and miscellaneous examples of a good type, and answers are added.

A student who has mastered the contents of this book should have a sound knowledge of useful working mathematics: even if his knowledge of principles is not very great, the method of teaching should have put him into a position in which generalization has become a habit.

An Introduction to the use of Common Logarithms. By James Rodger, B.A. Pp. 40. 1s. (Blackie Ltd.)

This little work is written in a chatty style, and should prove of service to those for whom it is intended,—evening and technical students of moderate preliminary education. The several rules are explained in a simple manner; the bugbear to this class of student, manipulation of negative characteristics, receiving ample treatment.

An Introduction to Geometry. By E. O. Taylor. Pp. 132. 1s. 6d. 1912. (Clarendon Press.)

The treatment of the subject in this book is, in the main, according to the recommendations and suggestions of the Board of Education, as set forth in Circular 771. An abundance of concrete illustrations is given and the use of home-made apparatus is encouraged. Numerous sets of test questions are scattered throughout the text. The whole tone of the book is one to encourage interest and imagination, and it should form a useful text-book for a first year's course in Geometry.

Elementary Geometry. By A. E. Layng. Pp. 247. 3s. 1912. (Murray.)

The book opens with the consideration of forms of the simpler solids, direction and angle, and at once proceeds to the demonstration of the fundamental theorems on angles, parallels and angles of a polygon. This is followed by a section on the construction of a specified triangle and congruence, preparing the student for theorems on congruence of triangles, and the properties of a parallelogram. Then follows a section on fundamental constructions, with the proofs left to the student. This method is followed to a certain extent throughout, the practical Geometry being skilfully interwoven with the theory. Teachers who think that the time lavished latterly on practical Geometry, with instruments and without reasoning, is a waste of time so far as education is concerned, might do worse than give this text-book a good trial. A Teacher's copy, with answers, can be obtained for 3s. 6d.

Elements of Arithmetic. Part II. By Hall & Stevens. Pp. 289. 2s. 1912. (Macmillan.)

A good selection of exercises taken from the Authors' *School Arithmetic*; they embrace square root, areas and volumes, ratio and proportion, percentages, interest and discount, stocks and shares, continued fractions and logarithms. There is also a set on graphs preceded by a full explanation; notes of method, etc., are freely given. A valuable addition to a teacher's class-room library. J. M. Child.

Notions sur les six coordonnées homogènes d'une droite. By A. Sefarian. Pp. 99. 1910. (Lausanne.)

If a unit force has components $X\,Y\,Z$ along and moments $L\,M\,N$ about axes Ox, Oy, Oz, the line of action of the force is completely determined by these six magnitudes, which are connected by the identical relation $LX+MY+NZ=0$. M. Sefarian works out several properties by means of these coordinates, and defining a linear complex of lines, obtains properties of the types considered, for example, in Routh's *Statics*, vol. i. pp. 208 *et seq.* C. S. J.

THE LIBRARY.

The Librarian acknowledges, with many thanks, the receipt of thirty volumes from Mr. Levett (one of our original members), who has also sent five bound volumes of Mathesis, a copy of the *First Annual Report* from Mr. Philpott, and of three volumes from Mr. Gallatly.

The Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2	copies of *Gazette*	No. 2 (very important).
2 or 3	copies of Annual Report	No. 11 (very important).
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1 copy	„	Nos. 1, 2.

NOTICE.

The promised brochure containing the reproduction of a certain number of the beautiful 'Geometrical Anaglyphs' shown by MM. Vuibert at the recent Mathematical Congress has now appeared, under the title *Les Anaglyphes géométriques.* (Librairie Vuibert, 63 Boulevard Saint Germain, Paris.)

A more detailed notice will be given later on.

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